Confinement of Quarks through π_1 Topological objects

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- The mechanism of quark confinement is one of the big remaining theoretical questions in QCD
- It is a non-perturbative feature – it cannot be analysed in perturbation theory
- At low temperatures and densities, there are no free quarks, only mesons, Baryons, . . .
- The strong interaction provides a strong linear potential between quarks at intermediate distances
- Quarks have the property of colour (red, blue, green)
- Mesons, Baryons, . . . are colourless – a Meson consists of a red quark and an anti-red quark
- Here we focus on confinement of Mesons
- Two questions need answering:
  1. Why is there a linear force between quarks at intermediate distances?
  2. Why are mesons colourless?

- We are primarily interested in QCD,
- QCD is a non-Abelian gauged quantum field theory
- The Guage Bosons are described by objects which satisfy the Lie algebra of SU(3)
- The phenomenon of confinement is also present in 'SU(2) QCD'
- SU(2) QCD is simpler but has the same underlying principles
- QCD describes the interactions between two types of object:
  - Fermion fields ψ, the quarks and anti-quarks
  - Gauge fields A_μ, the eight types of gluons

- One way of thinking about the interaction between quarks and gluons is to put QCD on a space time lattice
- The fermion fields exist on the sites of the lattice
- Their positions in each direction are an integer multiple of a lattice spacing a
- To preserve gauge invariance, whenever we translate a fermion field, we multiply it by the gauge link $U_\mu(x_1, x_2)$

$$\psi(x_1) \underset{\text{translation}}{\longrightarrow} U_\mu(x_1, x_1 + a\hat{\mu})\psi(x_1 + a\hat{\mu}))$$

$$U_\mu(x_1, x_1 + a\hat{\mu}) = P\left[e^{ig\int_{x_1}^{x_1+a\hat{\mu}} A_\mu dx^\mu}\right]$$

- A simple covariant derivative operator is defined as

$$\nabla_\mu\psi(x) = \lim_{a\to 0}\frac{1}{a}\left[U_\mu(x_1, x_1 + a\hat{\mu})\psi(x + a\hat{\mu}) - \psi(x)\right]$$

- The gauge field is defined as

$$igA_\mu(x) = U^\dagger_\mu(:, x)\partial^{(x)}_\mu U_\mu(:, x)$$

- U is an $SU(N)$ matrix ($N > 1$),
- A is a $N \times N$ Hermitian traceless matrix
- $A_\mu = A^a_\mu \lambda^a$ where λ^a are the Pauli or Gell-Mann matrices
- The field is non-Abelian, so different Us don't commute with each other

- Consider a curve C, a $R \times T$ rectangle in the xt plane
- Split the curve into infinitesimal segments (σ = distance around curve)
- Take the product of Us for each segment around the curve
- Take the limit as the length of the segments $\delta\sigma \to 0$

$$W_L[C,U] = \lim_{\delta\sigma\to 0} P\left[\prod_{\sigma\in C} U_{\mu[\sigma]}(x[\sigma])\right]$$

- In Euclidean space time, this represents the force $V(R)$ between a pair of static quarks propagating through time

$$\langle \text{tr}\, W_L[C,U]\rangle = A_0 e^{-V(R)T} + \ldots$$

- If $V(R)T \propto$ area of the loop $\Rightarrow$ a linear potential
- If each individual Wilson Loop, W_L, scales like $e^{i\chi}$, where the mean $\langle|\chi|\rangle \propto$ the area of the loop $\Rightarrow$ a linear potential.

- How are we to interpret an area law Wilson Loop?
- Consider an Abelian gauge theory
- No need for path ordering:

$$W_L[C,U] = \prod_{\sigma \in C} U_{\mu[\sigma]}(x[\sigma]) = e^{ig \oint A_\mu dx^\mu} = e^{ig \int d^2 S^{\mu\nu} F_{\mu\nu}}$$

$$S = \text{Surface Bound by } C \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Suppose we have lines of large electric flux
- χ counts how many lines of flux pass through the surface
- Number of flux lines $\propto$ Area of Loop $\Rightarrow$ Area law

- Confinement is non-perturbative
- Why does perturbation theory fail?
- Perturbation theory gets the QCD vacuum wrong
- The perturbative expansion is around $gA = 0$
- That is not the actual QCD vacuum
- In practice, the QCD vacuum contains many objects or lumps
- These are characterised by a non-zero topological index
- Proposed topological objects include Monopoles, Instantons, Vortices, Membranes, ...
- It is plausible that these objects play a role in quark confinement

- Currently two dominant models of confinement:
  1. The Dual Meissner effect: An analogue to the type II superconductor, with magnetic monopoles playing the role of the electrons, pushing the electric field into 1D flux tubes
  2. The Vortex theory: There exist one dimensional topological objects in the QCD Vacuum; these form long lines of electric flux
- In this talk, I shall discuss a third model
- The topological objects of interest are neither monopoles or vortices
- But first, we need to figure out how to apply this intuitive picture for a non-abelian theory

- We make use of the gauge-invariant Abelian decomposition
- We start by choosing a field $\theta(x) \in SU(N)$ (or $\theta_x \in SU(N)$)
- We require that this field is differentiable
- We then construct $n_a = \theta\lambda_a\theta^\dagger$
- We select the Abelian directions $n_j \equiv n_3, n_8, \ldots$
- We choose fields $\hat{A}_\mu$ and X_μ so that

$$A_\mu = \hat{A}_\mu + X_\mu \qquad D_\mu[\hat{A}]n_j = 0 \qquad \text{tr}(n_j X_\mu) = 0$$

- $\hat{A}$ represents colour-neutral gluons; X coloured gluons
- This has a known and unique solution,

$$\hat{A}_\mu = \frac{1}{2} n_j \text{tr}(n_j A_\mu) + \frac{i}{4g}[n_j, \partial_\mu n_j]$$

$$F_{\mu\nu}[\hat{A}] = \frac{n_j}{2}\left[\partial_\mu \text{tr}(n_j A_\nu) - \partial_\nu \text{tr}(n_j A_\mu)\right] + \frac{i}{8g} n_j \text{tr}(n_j[\partial_\mu n_k, \partial_\nu n_k]).$$

- On the lattice, we can write this as

$$U_{\mu,x} = \hat{X}_{\mu,x}\hat{U}_{\mu,x},$$
$$\hat{U}_{\mu,x} n_{j,x+\hat{\mu}} \hat{U}^\dagger_{\mu,x} - n_{j,x} = 0 \qquad \text{tr}(n_{j,x}(\hat{X}_{\mu,x} - \hat{X}^\dagger_{\mu,x})) = 0$$

- Choose solution where tr$\hat{X}_\mu$ is maximised
- $U \equiv$ standard gauge link constructed from A ($U \sim e^{igA_\mu \delta x_\mu}$)
- $\hat{U} \equiv$ gauge link constructed from $\hat{A}$ ($\hat{U} \sim e^{ig\hat{A}_\mu \delta x_\mu}$)
- $\hat{X}$ corresponds to X ($\hat{X} \sim e^{igX_\mu \delta x_\mu}$)
- The gauge transformations are ($\Lambda_x \in \text{SU}(N)$)

$$U_{\mu,x} \to \Lambda_x U_{\mu,x+\hat{\mu}} \Lambda^\dagger_{x+\hat{\mu}} \qquad \theta_x \to \Lambda_x \theta_x,$$
$$\hat{U}_{\mu,x} \to \Lambda_x \hat{U}_{\mu,x+\hat{\mu}} \Lambda^\dagger_{x+\hat{\mu}} \qquad \hat{X}_{\mu,x} \to \Lambda_x \hat{X}_{\mu,x} \Lambda^\dagger_x$$

- Paths of gauge links constructed from $\hat{U}$ are gauge covariant.

- So how do we choose θ_x?
- We will extract the static potential from the Wilson Loop
- We choose θ_x so that along the Wilson Loop $\hat{U}_\mu = U_\mu$.
- This θ_x contains the eigenvectors of the Wilson Loop operator starting and ending at position x.
- This guarantees that the Wilson Loop for the restricted field is identical to the Wilson Loop for the non-Abelian field.
- Furthermore,

$$\hat{U}_\mu(x) n_{j,x+\hat{\mu}} \hat{U}_\mu(x)^\dagger - n_{j,x} = 0 \quad \Leftrightarrow \quad [\theta_x^\dagger \hat{U}_{\mu,x} \theta_{x+\hat{\mu}}, \lambda_j] = 0,$$

- $\theta_x^\dagger \hat{U}_{\mu,x} \theta_{x+\hat{\mu}}$ is Abelian and gauge invariant

- We therefore take our Wilson Loop operator W_L

$$W_L[C,U] = W_L[C,\hat{U}] = \lim_{\delta\sigma\to 0} \prod_{\sigma\in C} \hat{U}_{\mu[\sigma]}(x[\sigma])$$

- Insert a pair of θ fields between each gauge link

$$W_L[C,U] = \lim_{\delta\sigma\to 0} \theta \left[\prod_{\sigma\in C} \theta^\dagger_{x[\sigma]} \hat{U}_{\mu[\sigma]}(x[\sigma]) \theta_{x[\sigma+\delta\sigma]} \right] \theta^\dagger$$

- $\theta^\dagger_x \hat{U}_{\mu,x} \theta_{x+\hat{\mu}} = e^{i\hat{u}^j_\mu \lambda_j \delta x_\mu}$

$$W_L[C,U] = \theta \left[e^{ig\oint \lambda_j \hat{u}^j_\mu dx_\mu} \right] \theta^\dagger$$

- We have removed the path ordering.
- The coloured field X does not contribute to confinement – mesons are colour-neutral.

$$\hat{A}_\mu = \frac{n_j}{2}\mathrm{tr}(n_j A_\mu) + \frac{i}{4g}[n_j, \partial_\mu n_j] = \frac{n_j}{2}\mathrm{tr}(n_j A_\mu - i\lambda_j \theta\partial_\mu\theta^\dagger) + \frac{i}{g}\theta\partial_\mu\theta^\dagger$$

$$F_{\mu\nu}[\hat{A}] = \frac{n_j}{2}\left[\partial_\mu \mathrm{tr}(n_j A_\nu) - \partial_\nu \mathrm{tr}(n_j A_\mu)\right] + \frac{i}{8g} n_j \mathrm{tr}(n_j[\partial_\mu n_k, \partial_\nu n_k])$$

- Both $F_{\mu\nu}[\hat{A}]$ and $\hat{A}_\mu$ depend on two terms:
  1. A function of both A_μ and θ (the Maxwell term)
  2. A function of θ alone (the Topological term)
     (Topological field strength $= H^{3,8}_{\mu,\nu}$).
- Parametrise the SU(2) θ in terms of a, c, d

$$\theta = e^{ia\phi}e^{id\lambda_3}, \quad \phi = \begin{pmatrix} 0 & e^{ic} \\ e^{-ic} & 0 \end{pmatrix}, \quad \bar{\phi} = \begin{pmatrix} 0 & ie^{ic} \\ -ie^{-ic} & 0 \end{pmatrix}$$

- d makes no contribution ($n_3 = \theta\lambda_3\theta^\dagger$) – fix it to zero
- $\theta^\dagger\partial_\mu\theta = -i\lambda_3 \sin^2 a\partial_\mu c + i\bar{\phi}\sin 2a\partial_\mu c + i\phi\partial_\mu a$
- $\mathrm{tr}(n_3[\partial_\mu n_3, \partial_\nu n_3]) = \partial_\mu a\partial_\nu c - \partial_\nu a\partial_\mu c$.

$$\theta = \begin{pmatrix} \cos a & i\sin a e^{ic} \\ i \sin a e^{-ic} & \cos a \end{pmatrix}$$

- c is ambiguous at $a = 0$ or $a = \pi/2$,
- Can have non-zero winding number around these points without a non-differentiability in θ
- We want to map a and c to E^4; (Wilson Loop in xt plane)

$$(t, x, y, z) = r(\cos\psi_3, \sin\psi_3\cos\psi_2, \sin\psi_3\sin\psi_2\cos\psi_1, \sin\psi_3\sin\psi_2\sin\psi_1).$$

- There are two types of topological object available
  1. The Wang-Yu magnetic Monopole (π_2 topology): $a = \psi_1/2, c = \nu_{WY}\psi_2$.
  2. Another object (π_1 topology) with $a(r, \psi), c = \nu_T\psi_3$, Appears at $a \sim 0$ or $a \sim \pi/2$.
- ν_{WY} and ν_T are integer winding numbers.
- Both winding numbers are invariant under continuous gauge transformations and deformations of the gauge field.

- We apply Stoke's theorem to our Abelian representation of the Wilson Loop
  1. A surface integral over the continuous part of $F_{\mu\nu}[\hat{A}]$
  2. Line integrals around each topological singularity
- This line integral resembles $\oint dx_\mu (\sin^2 a)\partial_\mu c$.
- It is proportional to the winding number ν_T (the monopoles do not directly contribute).
- Since the number of these objects is proportional to the area, we expect an area law scaling for the Wilson Loop.

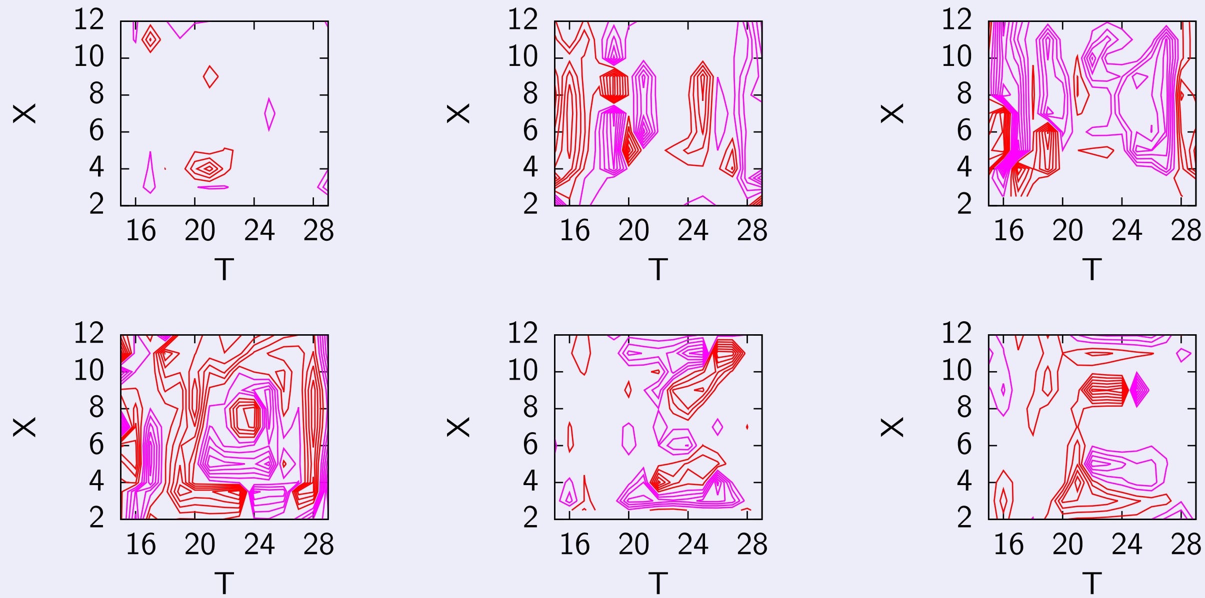
- It is easy to calculate the topological field strength surrounding each of these objects.
- We characterise $H^{3,8}_{\mu\nu}$ in terms of 'Electric' and 'Magnetic' fields.

1. **Monopoles**: Nothing in the E_x field
   1-D lines of high field strength in B_x, B_y and B_z parallel to the T axis; or
   1-D lines in B_x, E_y and E_z parallel to the X axis

2. π_1 **objects**: 0-D Points in the E_x field, accompanied by some of
   1-D lines in B_x, B_y and B_z parallel to the T axis; and/or
   1-D lines in B_x, E_y and E_z parallel to the X axis

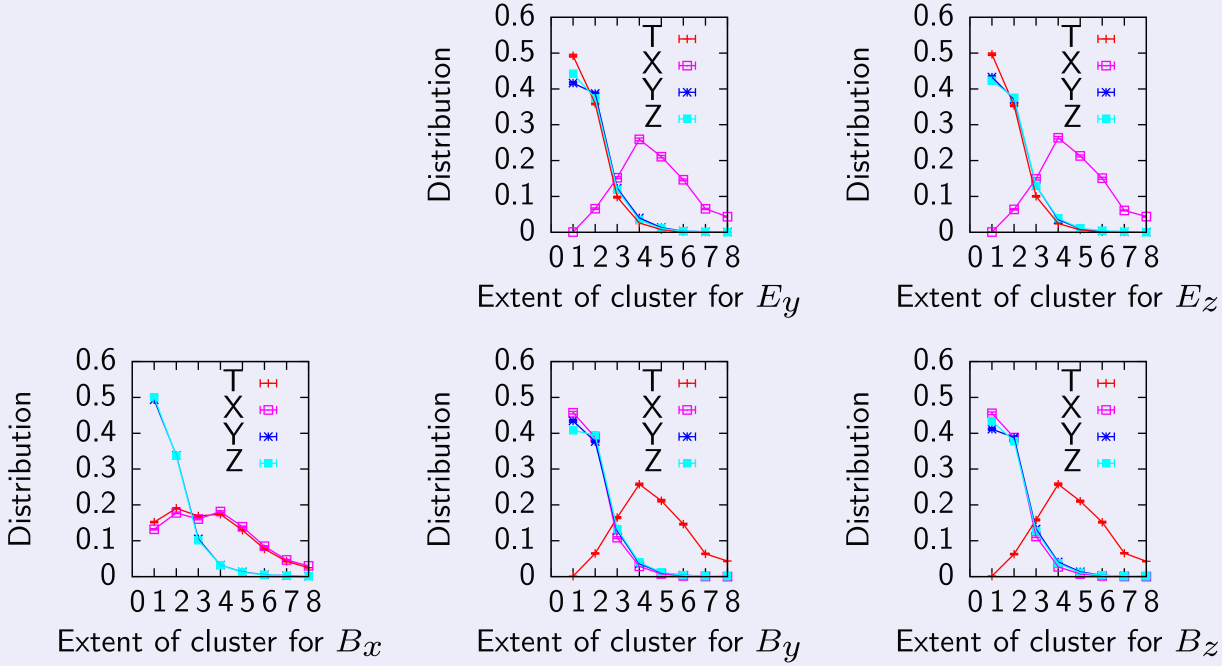
[All plots on a $16^3 \times 32$ $\beta 8.52$ TILW gauge action quenched ensemble, $a \sim 0.08\mathrm{fm}$].

The X (left) Y (middle) and Z (right) components of the Electric (top) and Magnetic (bottom) Abelian Field Strengths



The plots show the field strength $H^{8}_{\mu\nu}$ (red and purple contour lines) on a slice of the lattice in the XT plane.

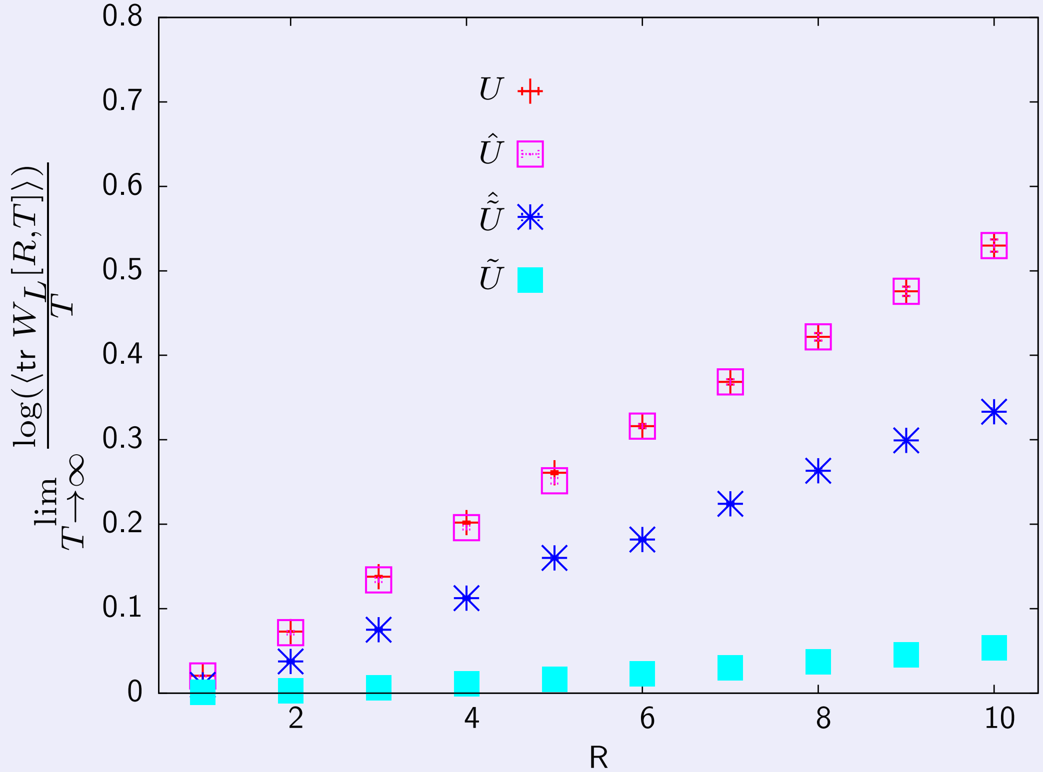
These plots show the extent in each spatial direction of structures of connected high field strength for the $H^8_{\mu\nu}$ field.



The location of the strings are correlated with the spatial location of the points in the E_x field.

So it seems like the topological vacuum does indeed contain both monopoles and the π_1 topological objects.

- So does the topological part dominate the string tension ρ calculated from the restricted Abelian field $\hat{A}$?
- We construct a highly smoothed gauge field $\tilde{A}_\mu$, respects gauge invariance but is not confining (using Stout Smearing)
- Apply the Abelian decomposition, using original $\theta \Rightarrow \hat{\tilde{A}}_\mu$
- Calculate the string tension for this field, and compare it to the results from the U and $\hat{U}$
- This allows us to extract the topological part of the string tension gauge covariantly
- Interested in the ratio of the topological string tension to the full string tension
- If this ratio is one, then confinement is fully accounted for by topological objects including those we are studying
- Our lattice results (still not finalised) show a strong dependence on the lattice spacing
- Volume dependence (not shown here) is negligible



lattice spacing (fm)	0.14	0.11	0.08
U string tension	0.0934(49)	0.0740(17)	0.0501(13)
$\hat{U}$ string tension	0.1044(72)	0.0790(30)	0.0521(17)
Topological string tension	0.0279(20)	0.0243(15)	0.0274(10)
Ratio	0.267(27)	0.308(22)	0.526(26)

Conclusions

- We have proposed a new model which explains:
  - The area Law scaling of the Wilson Loop
  - Why Mesons are colour neutral
- The field strength seems to back up the model – we see the objects we expect
- However, we cannot yet say whether or not the topological part dominates the string tension
- Thus we cannot say how much of the confining potential comes from the objects studied here
- Further Theoretical and Numerical work is still needed to fully explore our model